FUN WITH RREF

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As I noted, some background in matrix algebra is required for this discussion namely, material from my text ALAMA in Chapters 1 (Linear Systems of Equations) and 2 (Matrix Algebra). Throughout, $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$ is an $m \times n$ matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, $F = [\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n]$ is the RREF of A and any time I use parentheses I mean a column (saves space). We have these basic facts:

(1) If the pivot columns of F in order are $\mathbf{f}_{i_1}, \mathbf{f}_{i_2}, \ldots, \mathbf{f}_{i_r}$ and $I_r = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_r]$ is the $r \times r$ identity matrix, then $[\mathbf{f}_{i_1}, \mathbf{f}_{i_2}, \ldots, \mathbf{f}_{i_r}] = \begin{bmatrix} I_r \\ \mathbf{0}_{n-r,r} \end{bmatrix}$, where $\mathbf{0}_{n-r,r}$ is an $(n-r) \times r$ matrix of zeros.

(2) If $\mathbf{f}_i = (f_1, f_2, \ldots, f_p, 0, \ldots, 0)$ is a non-pivot column of F with preceding pivot columns $\mathbf{f}_{i_1}, \mathbf{f}_{i_2}, \ldots, \mathbf{f}_{i_p}$, then

$$\mathbf{f}_i = f_1\mathbf{f}_{i_1} + f_2\mathbf{f}_{i_2} + \cdots + f_p\mathbf{f}_{i_p}.$$

(3) Each elementary row operation used to find the RREF of $m \times n$ matrix A can be accomplished by left multiplication by an $m \times m$ invertible matrix which is formed by applying the row operation to identity matrix I_m. Specifically: Interchanging rows i and j yields elementary matrix E_{ij}, adding c times row j to row i yields $E_{ij}(c)$, and multiplying row i by the constant $c \neq 0$ yields $E_i(c)$. Similar facts hold for column operations via right multiplication by suitably transformed identity matrices.

(4) Let V be the product of the elementary matrices of Fact 3 that yield F from A in the correct order, so that $F = VA$ or, equivalently, $A = V^{-1}F$. In terms of columns, this is equivalent to $\mathbf{a}_i = V^{-1}\mathbf{f}_i$ for $i = 1, \ldots, n$.

(5) Form the sequence of column numbers of the pivot element in the RREF of the $m \times n$ matrix $A = [a_{i,j}]$ in increasing order, say $j_1 < j_2 < \cdots < j_r$. Likewise, form the sequence of independent pivot row indices of A in increasing order, say $i_1 < i_2 < \cdots < i_r$. The $r \times r$ matrix $W = [a_{i_r, j_s}]_{r,s=1,\ldots,r}$ formed from A by collection entries with the above increasing row and column numbers is called the *intersection* of the independent rows and columns of A. Since its RREF is just I_r, W is certainly invertible.

With these facts under our belt, proofs of these theorems can be established:

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Theorem 1. *(CR Factorization) Suppose that A is an $m\times n$ matrix, C is the $m\times r$ matrix whose columns are the independent columns of A in the order prescribed by the pivot columns of its RREF and R is the $r\times n$ matrix formed by deleting the zero rows of the RREF of A. Then $A = CR$.*

Proof. Let $C = [\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r] = [\mathbf{a}_{i_1}, \mathbf{a}_{i_2}, \dots, \mathbf{a}_{i_r}]$ where the column $\mathbf{a}_{i_j}$ is the pivot column that is transformed to $\mathbf{e}_j$ in the RREF of A ($\mathbf{e}_j$ as in Fact 1) and $R = [\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n]$ (which is just the result of deleting the zero rows of F). Then

$$CR = [C\mathbf{r}_1, C\mathbf{r}_2, \dots, C\mathbf{r}_n].$$

If $\mathbf{r}_i$ is the jth pivot column, the $\mathbf{r}_i = \mathbf{e}_j$ and $C\mathbf{r}_i = C\mathbf{e}_j = \mathbf{c}_j = \mathbf{a}_{i_j}$, which is the column of A that is transformed into $\mathbf{e}_j$ in its RREF.

On the other hand, if $\mathbf{r}_i$ is the non-pivot column as in Fact 2, then

$$C\mathbf{r}_i = f_1\mathbf{a}_{i_1} + f_2\mathbf{a}_{i_2} + \cdots + f_p\mathbf{a}_{i_p}.$$

Multiply the equation of Fact 2 by V^{-1} to obtain that the right-hand side of the above equation is just $V^{-1}\mathbf{f}_i = \mathbf{a}_i$, so that in all cases, $C\mathbf{r}_i = \mathbf{a}_i$. Hence, $CR = A$. □

Theorem 2. *(CWB factorization) Suppose that A is an $m\times n$ matrix, C is the $m\times r$ matrix whose columns are the independent columns of A in the order prescribed by the pivot columns of the RREF of A, W is the intersection of the independent rows and columns of A, and B is the $r\times n$ matrix formed by the independent rows of A in the prescribed order. Then $A = CW^{-1}B$.*

Proof. In the notation of Theorem 1, $A = CR$. Hence, it suffices to prove that $W^{-1}B = R$. Let $m\times m$ P and $n\times n$ Q be permutation matrices that move the independent rows and columns to initial positions in the prescribed order. Then

$$PAQ = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = PCRQ.$$

On the other hand, since the independent columns of A form C, left multiplication of C by P moves the independent rows up to the top in order. Also, right multiplication of R by Q moves the pivot columns to the left in order. Hence,

$$PAQ = (PC)(RQ) = \begin{bmatrix} W \\ Y \end{bmatrix} \begin{bmatrix} I_r & V \end{bmatrix} = \begin{bmatrix} W & WV \\ Y & YV \end{bmatrix},$$

so $\begin{bmatrix} W & WV \end{bmatrix}$ consists of the independent rows of A with columns permuted by Q, $B = \begin{bmatrix} W & WV \end{bmatrix} Q^{-1}$ consists of the independent rows of A and $\begin{bmatrix} I_r & V \end{bmatrix}$ is just R with colmns permuted by Q. It follows that

$$R = \begin{bmatrix} I_r & V \end{bmatrix} Q^{-1} = W^{-1} \begin{bmatrix} W & WV \end{bmatrix} Q^{-1} = W^{-1}B,$$

which completes the proof. □